MZ-TH/93-33

Two Different Formulations of the Heavy Quark Effective Theory[1]

S. Balk[2], A. Ilakovac[3], J. G. Körner[3] and D. Pirjol[2]

Johannes Gutenberg-Universität, Institut für Physik,
Staudinger Weg 7, D-55099 Mainz, Germany

Abstract: We point out that there exist two different formulations of the Heavy Quark Effective Theory (HQET). The one formulation of HQET was mostly developed at Harvard and involves the use of the equation of motion to eliminate the small components of the heavy quark field. The second formulation, developed in Mainz, involves a series of Foldy-Wouthuysen-type field transformations which diagonalizes the heavy quark Lagrangian in terms of an effective quark and antiquark sector. Starting at $\mathrm{O}(1/m_Q^2)$ the two formulations are different in that their effective Lagrangians, their effective currents, and their effective wave functions differ. However, when these three differences are properly taken into account, the two alternative formulations lead to identical transition or S-matrix elements. This is demonstrated in an explicit example at $\mathrm{O}(1/m_Q^2)$. We point to an essential difficulty of the Harvard HQET in that the Harvard effective fields are not properly normalized starting at order $O(1/m_Q^2)$. We provide explicit higher order expressions for the effective fields and the Lagrangian in the Mainz approach, and write down an $\mathrm{O}(1/m_Q^3)$ nonabelian version of the Pauli equation for the heavy quark effective field.

---

[1] Invited talk given by J.G.Körner at the conference "Hadron Structure 93" in Banska Stiavnica (Slovakia) (to be published in the Proceedings) and at the "XXVII International Symposium on the Theory of Elementary Particles" in Wendisch-Rietz (Germany) (1993)

[2] Supported by the Graduiertenkolleg "Teilchenphysik", Mainz

[3] Supported in part by the BMFT, Federal Republic of Germany, under contract 06MZ730

Let us get straight into the heart of the matter of what we want to discuss in this talk by stating that there exist two different formulations of the Heavy Quark Effective Theory (HQET) which will be referred to as the Harvard HQET [1, 2] and the Mainz HQET [3, 4].

The HQET that is most widely in use was mostly developed at Harvard [1, 2]. It is for this reason that we shall refer to this version as the Harvard HQET. In the Harvard approach one first extracts the mass phase from the heavy quark field $\psi_Q(x)$ and then splits up the residual field into its "large component" and "small component" pieces $h(x)$ and $H(x)$, respectively. Accordingly one writes

$$\psi_Q(x) = e^{-im_Q v\cdot x}(h(x) + H(x)) \tag{1}$$

where

$$\not{v}h(x) = h(x) \tag{2}$$

and

$$\not{v}H(x) = -H(x) \tag{3}$$

and where v_μ is the four-velocity of the heavy quark, $v_\mu = p_\mu/m_Q$. Unfortunately the nomenclature in terms of the "large" component field $h(x)$ and the "small" component field $H(x)$ has been somewhat tangled up in the course of developing the Harvard HQET. In the following we shall drop reference to the x-dependence in the fields.

In the next step one takes "one-half" of the full equation of motion of QCD, i.e.

$$(i\not{D} - m_Q)\psi_Q = 0 \tag{4}$$

by applying the small component projector $(1 - \not{v})/2$ to Eq. (4):

$$(iv\cdot D + 2m_Q)H = i\not{D}_\perp h \tag{5}$$

In Eq. (5) we have introduced the four-transverse component of the covariant derivative

$$D_\perp{}^\mu = D^\mu - v\cdot Dv^\mu \tag{6}$$

where the four-transversality is defined w.r.t the velocity v_μ. Eq. (5) can then be inverted to obtain

$$H = \frac{1}{(iv\cdot D + 2m_Q)} i\not{D}_\perp h \tag{7}$$

The small component field H can be seen to be related to the large component field h through a geometric series in the inverse power of the heavy mass. The Harvard approach consists in eliminating the small component field out of the theory via Eq. (7). Before we proceed any further there are two asides that we want to embroider on.

First, there is the concept of the four-transversality used in Eqs. (5) and (7). Note that any four-vector a_μ can be split into its components transverse and parallel to a given four-velocity v_μ according to

$$a^\mu = a_\perp{}^\mu + a_\parallel{}^\mu := (a^\mu - v\cdot a v^\mu) + (v\cdot a v^\mu) \tag{8}$$

The concept of four-transversality is an important concept in the formalism of HQET as one is frequently referring to rest-frame objects where $v_\mu = (1,0,0,0)$ and where four-transverse vectors reduce to pure three-vectors ${a_\perp}^\mu \to \vec{a}$. This concerns the covariant derivative in Eq. (7) as well as the relative momenta ${k_\perp}^\mu = k^\mu - k\cdot v v^\mu$ and spin operators ${\gamma_\perp}^\mu = \gamma^\mu - \not{v} v^\mu$ that are needed in the construction of the spin wave-functions of HQET [5]. Explicit reference to the *four*-transversality is important in order not to get confused with the usual notion of transversality which refers to a three-transversality , i.e. $\vec{a} = \vec{a} - \vec{a}\cdot\hat{k}\hat{k}$ ($\hat{k}$ is a three-momentum of unit length). In fact, the concept of four-transversality should be quite familiar from QED when one is considering virtual photon exchange. In the rest frame of the virtual photon $\vec{q} = 0$ the conserved vector current amplitude $T_\mu = \langle b|J_\mu|a\rangle$, with $q^\mu\, T_\mu = 0$, reduces to a three-vector corresponding to the spin-one nature of the virtual photon. When expanding the conserved amplitude T_μ in terms of covariants in a general frame this has to be done in terms of four-transverse covariants e.g. ${p_\perp}^\mu = p^\mu - \frac{p\cdot q}{q^2}q^\mu$, where the four-transversality is defined w.r.t. the momentum q_μ of the photon.

The second aside concerns an alternative derivation of the Harvard HQET that was proposed in [2]. The authors of [2] employ functional integration techniques to integrate out the small component field H from the functional action. They then arrive at the same relation Eq. (7). That the equation of motion and functional integration derivations of HQET are entirely equivalent may be appreciated by considering the following simple example using the classical Lagrangian $L(x,y) = -\frac{x^2}{2} + yx$. If one wishes one may view x as being related to the small component field H and y as been related to the large component field h. Next consider the classical action integral

$$\int dy \int_{-\infty}^{\infty} dx e^{L(x,y)} = \int dy \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2 + yx} \tag{9}$$

The x-integration in (9) can be done by the usual "completion of the square" trick, i.e. by writing $-\frac{1}{2}x^2 + yx = -\frac{(x-y)^2}{2} + \frac{y^2}{2}$ One then arrives at

$$\int dy \int_{-\infty}^{\infty} dx e^{L(x,y)} = \sqrt{2\pi} \int dy e^{\frac{y^2}{2}} \tag{10}$$

After having integrated out the x-degree of freedom the new Lagrangian reads $L(y) = y^2/2$. The crucial observation is that the same Lagrangian is obtained by using the equation of motion obtained by variation of the $L(x,y)$ w.r.t. the x-degree of freedom, i.e. $\partial L(x,y)/\partial x = 0$. This then gives the equation of motion $-x + y = 0$. Substituting for x in the original Lagrangian $L(x,y)$ one obtains the same new Lagrangian $L(y)$ as before. We have chosen this simple illustration in order to make the point that the equation of motion and functional integration derivation of the Harvard HQET are entirely equivalent. This can be appreciated without having to go into the technicalities of the full functional integration approach presented in [2].

Returning to Eq. (7) one writes

$$H = \frac{1}{2m_Q}\frac{1}{(1+\frac{iv\cdot D}{2m_Q})} i\not{D}_\perp h = \frac{1}{2m_Q}\left(1 - \frac{iv\cdot D}{2m_Q} + \frac{(iv\cdot D)^2}{4m_Q^2} + \cdots\right) i\not{D}_\perp h \tag{11}$$

As mentioned before one then eliminates the small component field H by substituting Eq. (11) into the effective Lagrangian and current expressions. This defines the Harvard HQET in terms of a $1/m_Q$ geometric type series expansion of the effective Lagrangian and current in terms of the large component field h only.

In the Mainz HQET [3, 4] one uses a series of exponential Foldy-Wouthuysen-type field transformations which yield an exponential type series expansion for the effective Lagrangian and the effective current.

At this point we want to be rather suggestive and write down geometric and exponential series expansions for a given small parameter a. One has :

$$\text{geometric series:} \qquad \frac{1}{1-ia} = 1 + ia - a^2 \cdots \tag{12}$$

$$\text{exponential series:} \qquad e^{ia} = 1 + ia - \frac{1}{2}a^2 \cdots \tag{13}$$

The series have been arranged such that they coincide in the first order term. Although the two series would be an oversimplified representation of the two effective theories it still makes the point correctly: The Harvard and Mainz HQET's are different in that their effective Lagrangians and currents are different starting at $O(1/m_Q^2)$. However, despite the fact that the respective effective Lagrangians and currents are different, one expects the physics i.e. physical on shell matrix elements to be the same in both theories, if they are calculated correctly. That these issues are of immediate practical concern is being evidenced by the fact that $1/m_Q^2$ and even higher order corrections are presently being disscused in the literature [6, 7].

As a next step we want to explain in somewhat more detail how one derives the Mainz HQET. To start with let us briefly retrace the physics steps that lead one from the full QCD Lagrangian to the static heavy quark Lagrangian. This will first be done in a completely heuristic manner. Consider again the full QCD Lagrangian

$$\mathcal{L} = \bar{\psi}_Q (i\not{D} - m_Q)\psi_Q \tag{14}$$

The static approximation consists in neglecting the three-derivative in Eq. (14) relative to m_Q according to the expansion

$$E = \sqrt{m_Q^2 + \vec{p}^{\,2}} = m_Q \left(1 + \frac{\vec{p}^{\,2}}{2m_Q^2} + \cdots\right) \tag{15}$$

This can be achieved by an appropriate field redefinition $\psi_Q \to \psi'_Q$. This will then lead to

$$\text{Step I:} \qquad = \bar{\psi'}_Q (i\gamma_0 D_0 - m_Q)\psi'_Q \tag{16}$$

In the next step one shifts the energy scale $E \to E' = E - m_Q$ which can again be achieved by an appropriate field redefinition $\psi'_Q \to h_Q$. The Lagrangian now reads

$$\text{Step II:} \qquad = \bar{h}_Q (i\gamma_0 D_0) h_Q \tag{17}$$

If one chooses to work in terms of the heavy quark field $h^{(+)}$ only (with $h^{(+)} = \frac{1}{2}(1+\gamma_0)h_Q$) one then recovers the static Eichten-Hill Lagrangian $\mathcal{L} = \bar{h}^{(+)\dagger} iD_0 h^{(+)}$ [8].

The step-wise reduction of the QCD Lagrangian (14) to the final form (17) can be achieved by a series of Foldy-Wouthuysen-type field redefinitions which eventually yields the leading term result (17) as well as all higher dimension operators in the $1/m_Q$ expansion [5]. The first transformation is $(j = 1, 2, 3)$

$$\text{Step I:} \qquad \begin{aligned} \psi_Q &\rightarrow e^{i\gamma_j \vec{D}_j/2m_Q}\psi'_Q \\ \bar{\psi}_Q &\rightarrow \bar{\psi}'_Q e^{-i\gamma_j \overleftarrow{D}_j/2m_Q} \end{aligned} \tag{18}$$

where the arrow on the derivative indicates in which direction the derivative acts. The heavy quark Lagrangian now becomes

$$\mathcal{L} = \bar{\psi}'_Q(i\gamma_0 D_0 - m_Q)\psi'_Q + \sum_{k=1}^{\infty}\left(\frac{1}{m_Q}\right)^k \bar{\psi}'_Q \mathcal{O}_k \psi'_Q \tag{19}$$

giving a form of the Lagrangian which makes explicit the mass perturbations.

The second transformation that removes the heavy mass dependence in the first term of (19) can be seen to be given by

$$\text{Step II:} \qquad \begin{aligned} \psi_Q &\rightarrow e^{-im_Q\gamma_0 t}h_Q \\ \bar{\psi}_Q &\rightarrow \bar{h}_Q e^{im_Q\gamma_0 t} \end{aligned} \tag{20}$$

However, in order not to bring in further numerator mass terms through the transformation (20) one needs to first block-diagonalize the higher order operators $\mathcal{O}_k$ appearing in (19). The block-diagonalization has to be done w.r.t. the upper and lower component of the heavy quark fields $\frac{1+\gamma_0}{2}\psi_Q$ and $\frac{1-\gamma_0}{2}\psi_Q$, respectively. This can be achieved order by order by splitting the operator $\mathcal{O}_k$ into two pieces $\mathcal{O}_k^c$ and $\mathcal{O}_k^a$ that commute and anticommute with γ_0, respectively. That is, one writes

$$\mathcal{O}_k = \mathcal{O}_k^c + \mathcal{O}_k^a \tag{21}$$

where

$$\mathcal{O}_k^{c,a} = \frac{1}{2}(\mathcal{O}_k \pm \gamma_0 \mathcal{O}_k \gamma_0) \tag{22}$$

and $[\mathcal{O}_k^c, \gamma_0] = 0$ and $\{\mathcal{O}_k^a, \gamma_0\} = 0$. The anticommuting operator $\mathcal{O}_k^a$ is then removed from that order of $(1/m_Q)^k$ by a further exponential type transformation. It is literally shifted down to become part of the higher dimension operator $\mathcal{O}_{k+1}$.

Note that one never introduces any further implicit time-derivatives ∂_0 through the field redefinitions in addition to the time-derivative in the lowest order term of the final Lagrangian (see (19)). Technically speaking, explicit higher order time-derivative terms do appear through the above field redefinitions. However, these higher order time-derivatives always come in as commutators that are related to the field strengh tensor (see Eq.(31))[3, 4]. This will become important when we discuss the Pauli equation later on.

Up to now we have remained in the rest frame $v_\mu = (1, 0, 0, 0)$ of the heavy quark (or antiquark) in order to stay as close as possible to the heuristic considerations that led us to the static Lagrangian (17). The field redefinitions that lead to the HQET

Lagrangian can in fact be done in any moving frame v_μ with $\vec{v} \neq 0$ by effecting the replacements

$$
\begin{aligned}
\gamma_0 D_0 &\to \not{v}\, v \cdot D := \not{D}_\parallel \\
-\gamma_j D_j &\to \not{D} - \not{v}\, v \cdot D := \not{D}_\perp
\end{aligned}
\tag{23}
$$

In the moving frame the heavy quark effective Lagrangian is then given by

$$
\begin{aligned}
\mathcal{L}^{KT}_{HQET} = \bar{h}_Q \{ & i\not{v}\, v \cdot D + \frac{1}{2m_Q}(-D^2 + (v \cdot D)^2 \\
& + \frac{i}{2} g \sigma_{\mu\nu} F^{\mu\nu} - i g \gamma_\mu \not{v} F^{\mu\nu} v_\nu) + \cdots \} h_Q
\end{aligned}
\tag{24}
$$

where $F^{\mu\nu}$ is the field strength tensor $F^{\mu\nu} = [D_\mu, D_\nu]$ ($D_\mu = \partial_\mu - igA^\mu$ and $A = A^a t^a$) and where we have kept terms up to $\mathcal{O}(1/m_Q)$ only. We chose to label the Mainz effective theory by the initials of the authors of [3]. Note that the HQET Lagrangian contains both heavy quark fields $h^{(+)}$ and heavy antiquark fields $h^{(-)}$ where $\not{v} h^{(\pm)} = \pm h^{(\pm)}$ and $h_Q = h^{(+)} + h^{(-)}$. Remember that this is different in the Harvard approach where the construction is done either in the quark sector or in the antiquark sector by adjusting the sign of the mass phase in Eq.(1). The quark and antiquark sector in the Mainz HQET are completely decoupled (at any order!) which is guaranteed by the very procedure of deriving the HQET Lagrangian. To be sure one can easily convince oneself that the nondiagonal contributions $\bar{h}^{(+)} \ldots h^{(-)}$ and $\bar{h}^{(-)} \ldots h^{(+)}$ induced by the first order terms $\sigma_{\mu\nu} F^{\mu\nu}$ and $\gamma_\mu \not{v} F^{\mu\nu} v_\nu$ in Eq.(24) cancel. On the other hand, the $\gamma_\mu \not{v} F^{\mu\nu} v_\nu$ contribution vanishes for the diagonal contributions $\bar{h}^{(+)} \cdots h^{(+)}$ and $h^{(-)} \cdots h^{(-)}$ and one thereby recovers the $\mathcal{O}(1/m_Q)$ Harvard HQET Lagrangian as promised before (see Eq. (26)).

As noted before the method of deriving the Mainz HQET proceeds step-wise order by order. This is an iterative procedure which lends itself to computer implementation. In fact we have written an efficient program in Mainz that achieves just this. For the fun of it the program has been run up to $\mathcal{O}(1/m_Q^{12})$ [9]. As a sample result we list the Mainz HQET Lagrangian for the heavy quark field $h^{(+)}$ up to order $\mathcal{O}(1/m_Q^5)$. One has

$$
\begin{aligned}
\mathcal{L}^{\mathcal{KT}}_{\mathcal{HQET}} = \bar{h}^{(+)} \Bigg[& i\not{D}_\parallel - \frac{1}{2m_Q} {\not{D}_\perp}^2 - \frac{i}{4m_Q^2} \left(\frac{1}{2} \not{D}_\parallel {\not{D}_\perp}^2 + \not{D}_\perp \not{D}_\parallel \not{D}_\perp + \frac{1}{2} {\not{D}_\perp}^2 \not{D}_\parallel \right) \\
& + \frac{1}{8m_Q^3} \left(\not{D}_\parallel \not{D}_\perp \not{D}_\parallel \not{D}_\perp + \not{D}_\parallel {\not{D}_\perp}^2 \not{D}_\parallel + \not{D}_\perp {\not{D}_\parallel}^2 \not{D}_\perp + \not{D}_\perp \not{D}_\parallel \not{D}_\perp \not{D}_\parallel + {\not{D}_\perp}^4 \right) \\
& + \frac{i}{16m_Q^4} \left(\frac{1}{2} {\not{D}_\parallel}^2 \not{D}_\perp \not{D}_\parallel \not{D}_\perp + \frac{1}{2} {\not{D}_\parallel}^2 {\not{D}_\perp}^2 \not{D}_\parallel + \frac{3}{2} \not{D}_\parallel \not{D}_\perp {\not{D}_\parallel}^2 \not{D}_\perp \right. \\
& + 2 \not{D}_\parallel \not{D}_\perp \not{D}_\parallel \not{D}_\perp \not{D}_\parallel + \frac{1}{2} \not{D}_\parallel {\not{D}_\perp}^2 {\not{D}_\parallel}^2 + \frac{11}{8} \not{D}_\parallel {\not{D}_\perp}^4 + \not{D}_\perp {\not{D}_\parallel}^3 \not{D}_\perp \\
& + \frac{3}{2} \not{D}_\perp {\not{D}_\parallel}^2 \not{D}_\perp \not{D}_\parallel + \frac{1}{2} \not{D}_\perp \not{D}_\parallel \not{D}_\perp {\not{D}_\parallel}^2 + \frac{3}{2} \not{D}_\perp \not{D}_\parallel {\not{D}_\perp}^3 \\
& \left. + \frac{1}{4} {\not{D}_\perp}^2 \not{D}_\parallel {\not{D}_\perp}^2 + \frac{3}{2} {\not{D}_\perp}^3 \not{D}_\parallel \not{D}_\perp + \frac{11}{8} {\not{D}_\perp}^4 \not{D}_\parallel \right)
\end{aligned}
$$

$$
-\frac{1}{32m_Q^5}\left(\not{D}_\parallel{}^2\not{D}_\perp\not{D}_\parallel{}^2\not{D}_\perp + 2\not{D}_\parallel{}^2\not{D}_\perp\not{D}_\parallel\not{D}_\perp\not{D}_\parallel + \not{D}_\parallel{}^2\not{D}_\perp{}^2\not{D}_\parallel{}^2\right.
$$

$$
+\frac{4}{3}\not{D}_\parallel{}^2\not{D}_\perp{}^4 + 2\not{D}_\parallel\not{D}_\perp\not{D}_\parallel{}^3\not{D}_\perp + 4\not{D}_\parallel\not{D}_\perp\not{D}_\parallel{}^2\not{D}_\perp\not{D}_\parallel
$$

$$
+2\not{D}_\parallel\not{D}_\perp\not{D}_\parallel\not{D}_\perp\not{D}_\parallel{}^2 + \frac{10}{3}\not{D}_\parallel\not{D}_\perp\not{D}_\parallel\not{D}_\perp{}^3 + \not{D}_\parallel\not{D}_\perp{}^2\not{D}_\parallel\not{D}_\perp{}^2
$$

$$
+\frac{5}{3}\not{D}_\parallel\not{D}_\perp{}^3\not{D}_\parallel\not{D}_\perp + \frac{4}{3}\not{D}_\parallel\not{D}_\perp{}^4\not{D}_\parallel + \not{D}_\perp\not{D}_\parallel{}^4\not{D}_\perp
$$

$$
+2\not{D}_\perp\not{D}_\parallel{}^3\not{D}_\perp\not{D}_\parallel + \not{D}_\perp\not{D}_\parallel{}^2\not{D}_\perp\not{D}_\parallel{}^2 + 2\not{D}_\perp\not{D}_\parallel{}^2\not{D}_\perp{}^3
$$

$$
+\not{D}_\perp\not{D}_\parallel\not{D}_\perp\not{D}_\parallel\not{D}_\perp{}^2 + 2\not{D}_\perp\not{D}_\parallel\not{D}_\perp{}^2\not{D}_\parallel\not{D}_\perp + \frac{5}{3}\not{D}_\perp\not{D}_\parallel\not{D}_\perp{}^3\not{D}_\parallel
$$

$$
+\not{D}_\perp{}^2\not{D}_\parallel\not{D}_\perp\not{D}_\parallel\not{D}_\perp + \not{D}_\perp{}^2\not{D}_\parallel\not{D}_\perp{}^2\not{D}_\parallel + 2\not{D}_\perp{}^3\not{D}_\parallel{}^2\not{D}_\perp
$$

$$
\left.\left.+\frac{10}{3}\not{D}_\perp{}^3\not{D}_\parallel\not{D}_\perp\not{D}_\parallel + \frac{4}{3}\not{D}_\perp{}^4\not{D}_\parallel{}^2 + 2\not{D}_\perp{}^6\right)\right] h^{(+)} \tag{25}
$$

As mentioned before, the time-derivative terms $\not{D}_\parallel$ appearing at $O(1/m_Q^2)$ and higher can all be rewritten in terms of the field strength tensor [3].

The ensuing discussion will be in terms of the heavy quark field $h^{(+)}$ only and, for the sake of convenience, we shall omit the label $(+)$ on the heavy quark field in the following. In order to pinpoint the differences in the Harvard and Mainz HQET's let us consider the heavy quark effective Lagrangians and the flavour-conserving heavy quark effective currents up to $\mathcal{O}(1/m_Q^2)$. As mentioned before it is at this order where they start to differ from one another. For the Harvard HQET one has

$$
\mathcal{L}_{HQET}^{Harvard} = \bar{h}\{iv\cdot D - \frac{1}{2m_Q}\not{D}_\perp{}^2 + \frac{i}{4m_Q^2}\not{D}_\perp v\cdot D\not{D}_\perp\}h \tag{26}
$$

$$
\begin{aligned}
J_{\mu,HQET}^{Harvard} = \bar{h}\{\Gamma_\mu &+ \frac{i}{2m_Q}(\Gamma_\mu\vec{\not{D}}_\perp - \overleftarrow{\not{D}}_\perp\Gamma_\mu) \\
&+ \frac{1}{4m_Q^2}(\Gamma_\mu v\cdot\vec{D}\vec{\not{D}}_\perp + \overleftarrow{\not{D}}_\perp\Gamma_\mu\vec{\not{D}}_\perp + \overleftarrow{\not{D}}_\perp v\cdot\overleftarrow{D}\Gamma_\mu)\}h
\end{aligned} \tag{27}
$$

The $\mathcal{O}(1/m_Q^2)$ difference terms for the Mainz HQET are given by [10]

$$
\mathcal{L}_{HQET}^{KT} = \mathcal{L}_{HQET}^{Harvard} + \frac{i}{4m_Q^2}\bar{h}\left(-\frac{1}{2}\not{D}_\perp{}^2 v\cdot D - \frac{1}{2}v\cdot D\not{D}_\perp{}^2\right)h \tag{28}
$$

$$
J_{\mu,HQET}^{KT} = J_{\mu,HQET}^{Harvard} - \frac{1}{4m_Q^2}\bar{h}\left(\frac{1}{2}\overleftarrow{\not{D}}_\perp^{\,2}\Gamma_\mu + \frac{1}{2}\Gamma_\mu\vec{\not{D}}_\perp^{\,2}\right)h \tag{29}
$$

There are a number of observations we want to make about the difference terms in Eqs. (28) and (29).

Let us first rewrite the difference term in the effective Lagrangian (28). One has

$$
-\frac{1}{2}\not{D}_\perp{}^2 v\cdot D - \frac{1}{2}v\cdot D\not{D}_\perp{}^2 = -\not{D}_\perp v\cdot D\not{D}_\perp - \frac{1}{2}\not{D}_\perp[\not{D}_\perp, v\cdot D] + \frac{1}{2}[\not{D}_\perp, v\cdot D]\not{D}_\perp \tag{30}
$$

When one looks at the time derivative term $v \cdot D$ in (30) (or alternatively D_0 in the rest frame) one can see that the first term on the r.h.s. of (30) cancels the time derivative term in the Harvard Lagrangian (26). The remaining two terms on the right hand side of (28) do not contain true time-derivatives since the commutator can be expressed in terms of the field strength tensor via

$$[\not{D}_\perp, v \cdot D] = -igF^{\mu\nu}\gamma_\mu v_\nu \tag{31}$$

The difference in the Mainz and Harvard Lagrangians lies in the fact that the time-derivative terms only appear at leading order in the Mainz Lagrangian but to all orders in the Harvard Lagrangian. This is exemplified at second order in the above example. As a consequence of this one can therefore quite easily write down Pauli equations to any desired order in the Mainz approach by using the equation of motion for the effective heavy quark field h in the rest frame of the heavy quark. For example, this has been done to $\mathcal{O}(1/m_Q^3)$ in [10]. The result reads

$$\begin{aligned} i\frac{\partial h}{\partial t} = \Bigg(& gA^{0a}t^a + \frac{(\vec{P} - g\vec{A}^a t^a)^2}{2m} - \frac{g}{2m}\vec{\sigma} \cdot \vec{B}^a t^a - \frac{g}{8m^2}(\mathrm{div}\vec{E}^a + f_{abc}\vec{A}^b \cdot \vec{E}^c)t^a \\ & - \frac{ig}{8m^2}\vec{\sigma} \cdot \mathrm{rot}\vec{E}^a t^a - \frac{ig^2}{8m^2} f_{abc}\vec{\sigma} \cdot (\vec{A}^b \times \vec{E}^c)t^a - \frac{g}{4m^2}\vec{\sigma} \cdot \vec{E}^a t^a \times (\vec{P} - g\vec{A}^b t^b) \\ & - \frac{1}{8m^3}[(\vec{P} - g\vec{A}^a t^a)^2 - g\vec{\sigma} \cdot \vec{B}^a t^a]^2 \\ & + \frac{g^2}{8m^3}\left[\vec{E}^a t^a \cdot \vec{E}^b t^b + \frac{i}{2} f_{abc}\vec{\sigma} \cdot (\vec{E}^a \times \vec{E}^b)t^c\right]\Bigg) h \end{aligned} \tag{32}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic colour fields. To our knowledge the Pauli equation has not been derived to this order before, let alone in the non-Abelian case. We hope that we have convinced the reader by now that the Mainz approach to HQET allows one to do so very efficiently.

As the next topic we want to discuss the calculation of physical matrix elements using the two different formulations of HQET. We again concentrate on the $\mathcal{O}(1/m_Q^2)$ contributions where the two formulations start to be different.

We have arranged the difference terms in eq. (28) and (29) in a rather suggestive manner by placing them one below another: except for the $v \cdot D$ factors in the Lagrangian the difference terms in the Lagrangian and in the current look quite similar to one another. In fact one finds that the differences in the effective Lagrangians "almost" cancel in physical transition matrix elements. They cancel except for some $\mathcal{O}(1/m_Q^2)$ contributions which can be absorbed in the definition of the $\mathcal{O}(1/m_Q^2)$ HQET wave functions or the interpolating field associated with the HQET state.

Let us explicitly demonstrate this cancelation in the flavour-conserving case by using the Feynman diagram language. One of the second order contributions of the Lagrangian difference term is drawn in Fig.1a. for $b \to b$ transitions. The $v \cdot D$ term from the "nonlocal" Lagrangian insertion can be seen to cancel against the inverse propagator $(v \cdot D)^{-1}$ adjoining it. In the coordinate space language the two vertices

in Fig.1a. are contracted to one point due to the relation $(iv \cdot D)\, S(x,y) = \delta(x-y)$. One thus remains with the effective "local" insertion drawn in Fig.1b. This will be canceled by the truly "local" $1/m_Q^2$ insertions from the effective current difference (29) which is not drawn here. Nonlocal insertions in which $v \cdot D$ operates on the outer heavy quark wave function vanish due to the equation of motion. There is one exception to this which is represented by the Feynman diagram drawn in Fig.1c. The $v \cdot D$ to the left of the nonlocal insertion is now canceled by the inverse propagator to the right of the local insertion and one remains with a $\mathcal{O}(1/m_Q^2)$ difference even for physical transitions. This difference corresponds to a $\mathcal{O}(1/m_Q^2)$ renormalization of the HQET wave function or, more precisely, the interpolating field that is associated with the HQET state. The upshot of the analysis is that the Mainz and Harvard formulations of HQET lead to identical physical matrix elements as long as one takes into account the differences in the $\mathcal{O}(1/m_Q^2)$ wave functions as drawn in Fig.2 in addition to the differences in the effective Lagrangians and effective currents. The same conclusion was reached in [10] using functional differentiation techniques.

However, in spite of the fact that the predictions for physical matrix elements are identical, there exists an essential difference between the two HQETs, in that the Harvard HQET lacks a correct normalization of the heavy quark field. We will illustrate the problem in the context of QCD with unconfined heavy quarks. When evaluating S-matrix elements for scattering processes by using the usual machinery of the reduction formalism, an assumption is implicitly made about the value of matrix elements like

$$\langle 0|h(x)|Q(p,s)\rangle = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m_Q}{E}}u(m_Q v,s)e^{-ip\cdot x}\,, \tag{33}$$

giving the wave-function of the ingoing or outgoing heavy quarks. This prejudice has its roots in the fact that such a relation holds indeed true (by construction), if the heavy quark field in the effective theory $h(x)$ is replaced by the QCD field $Q(x)$. However, this relation *is not true* if $h(x)$ is the heavy quark field in the Harvard HQET. The right-hand side has to be multiplied with a correction factor of the form

$$1 + \frac{p^2}{8m_Q^2} + \cdots \tag{34}$$

which deviates from unity by an amount proportional to the residual momentum p of the one–quark state $|Q(p,s)\rangle$. It is clear that the omission of this correction factor, such as it is done in the naive application of the Harvard HQET, will possibly lead to a wrong answer.

On the other hand, relation (33) is valid to any order in the Mainz HQET. This has been shown in [10] by proving that the heavy quark field in the Mainz effective theory is connected to the QCD field by a unitary transformation. The fact that the Harvard HQET field has a different normalization can be appreciated by noting that the two are related by

$$h^{KT}(x) = \left(1 + \frac{(\gamma \cdot D_\perp)^2}{8m_Q^2}\right) h^{Harvard}\,. \tag{35}$$

Another way of looking at the issue of the normalization of the heavy quark field is to examine the form of the heavy quark propagator in the vicinity of the one-particle pole. The requirement of correct normalization can be expressed by saying that the residue of this pole should be unity. That these two points of view are related can be seen from the form of the Källén–Lehmann representation for the heavy quark propagator [11]

$$S'_F(p) = \frac{1+\gamma\cdot v}{2}\int_0^\infty \mathrm{d}M^2 \frac{\rho(M^2)}{i(v\cdot p - M^2 + i\epsilon)} \tag{36}$$

where $\rho(M^2)$ is defined by a sum over all possible intermediate states

$$\rho(p^2)\delta_{\alpha\beta} = (2\pi)^3 \sum_n \delta^{(4)}(p_n - p)\langle 0|h_\alpha(0)|n\rangle\langle n|\bar{h}_\beta(0)|0\rangle\,. \tag{37}$$

If one then makes use of Eq.(33), the contribution of the one-quark intermediate state to the spectral function $\rho(M^2)$ can be seen to be as follows

$$\rho(M^2) \;\; = \;\; \delta(M^2)\,, \tag{38}$$

which gives a free field-like behaviour of the heavy quark propagator in the neighbourhood of the one-particle pole. On the other hand, in the Harvard HQET the residue of the propagator at the one-particle pole takes a different value (which is even a function of the residual momentum of the heavy quark). This has been shown through explicit calculation in a recent paper by A.Das [16].

The lack of maintaining the correct normalization of the field or wave functions by eliminating the "small" component via the "equation of motion" approach has long been known in the Abelian context of QED, where these issues were studied in connection with the Foldy-Wouthuysen transformation [12, 13, 14, 15]. Eliminating the "small" component naively leads to a nonhermitean Hamiltonian. In the case of an electron interacting with an external electric field, the lowest order manifestation of the non-hermiticity is an imaginary electric dipole moment. All these issues were very nicely discussed in the paper by A. Das [16].

Acknowledgement: We owe many thanks to George Thompson who generously shared with us his profound insights into Quantum Field Theory. He had always emphasized that the two methods of deriving the Harvard HQET discussed in the text are equivalent and, for illustration, provided us with the simple classical analogue presented in the text.

References

[1] H. Georgi, Phys. Lett. *B 240* (1990) 447; A. F. Falk, H. Georgi, B. Grinstein and M. B. Wise, Nucl. Phys. *B 343* (1990) 1;

[2] T. Mannel, W. Roberts and Z. Ryzak, Nucl. Phys *B 368* (1992) 204;

[3] J. G. Körner and G. Thompson, Phys. Lett. *B 264* (1991) 185;

[4] A.Das and V.S.Mathur, UR-1320 (1993)

[5] F. Hussain, J. G. Körner and G. Thompson, "Hadrons of arbitrary spin and heavy quark symmetry", Mainz Preprint *MZ-TH/93-23* (1993);

[6] A. F. Falk and M. Neubert, Phys. Rev. *D 47* (1993) 2965, 2982;

[7] G. P. Lepage, L. Magnea, C. Nakleh, U. Magnea and K. Hornbostel, Phys. Rev. *D 46* (1992) 4052;

[8] E. Eichten and B. Hill, Phys. Lett. *B 234* (1990) 511;

[9] A. Ilakovac, unpublished notes;

[10] S. Balk, J. G. Körner and D. Pirjol, "Heavy Quark Effective Theory at High Orders in $1/m$", Mainz Preprint *MZ-TH/93-13* (1993);

[11] J.D.Bjorken and S.D.Drell, Relativistic Quantum Fields, (McGraw-Hill, New York, 1968), Chap.16;

[12] R. P. Feynman, Quantum Electrodynamics, (W. A. Benjamin, New York, 1961;

[13] W. Pauli, Handbuch der Physik, Vol V/1, (Heidelberg, 1958);

[14] N. Fukuda, K. Sawada and M. Taketani Prog. Theor. Phys. *12* (1954) 156;

[15] S. Okubo, Prog. Theor. Phys. *12* (1954) 102, 603;

[16] A. Das, Rochester Preprint *UR-1325* (1993).

Figure Captions

Fig.1 $O(1/m_Q^2)$ contributions from the effective Lagrangian to the flavour-conserving $H_b(v) \rightarrow H_b(v')$ current transition. a) and b) show how the nonlocal contribution a) becomes a local current contribution b) through the contraction of two interaction points. In c) we show an external line contribution which can be absorbed into the definition of the HQET wave function.

Fig.2 $O(1/m_Q^2)$ difference of Harvard and Mainz HQET wave functions.